Amplitude Equations for Control

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Outline

- Vortex shedding in cylinder flow at $Re = 100$
- The Van der Pol oscillator: a model problem
  - One time-scale approach
  - Two times-scales approach
  - With forcing term
- The Ginzburg-Landau eq.
  - Forcing with $\omega_f \neq \omega_0$
  - Forcing with perturbation of linear operator ($\omega_f = 0$)
  - Forcing with ω_f close to ω_0
- The forced Navier-Stokes eq. with cylinder flow
  - Forcing with ω_f close to 0
  - Forcing with ω_f close to ω_0
  - Forcing with $\omega_f \neq \omega_0$
- Illustration for forced Navier-Stokes eq. with open-cavity flow

DNS simulation of cylinder flow at Re=100

Unforced simulation (Q1)

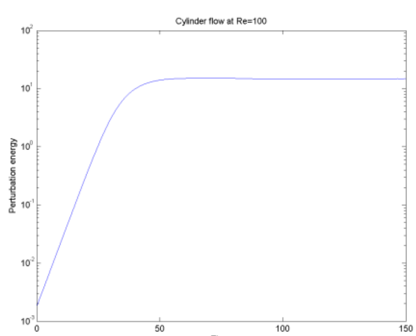
$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = 0$$

How does the system respond to harmonic forcing?

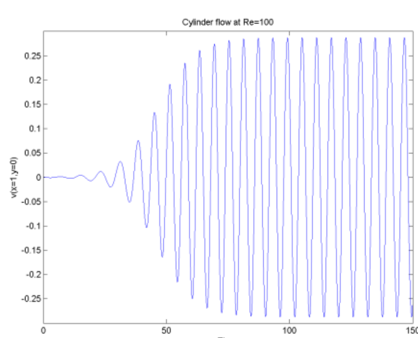
$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = \tilde{E}e^{i\omega_f}f + \text{c.c}$$

Influence of $\omega_f, f, \tilde{E}$?

DNS simulation of cylinder flow at Re=100



Energy vs Time



$v(x = 1, y = 0)$ vs Time

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The Van der Pol Oscillator: a model problem

ODE (similar to Navier-Stokes)

$$w'' + \omega_0^2 w = 2 \underbrace{\tilde{\delta}}_{\epsilon\delta} w' - w^2 w', \epsilon \ll 1, \delta = O(1)$$

$$w(0) = w_I, w'(0) = 0$$

Fixed point: $w = 0$

Stability of fixed point: $|w| \ll 1$

$$w'' + \omega_0^2 w = 2\epsilon\delta w'$$

Global modes:

$$w = e^{\lambda t}\hat{w} \Rightarrow \lambda^2 + \omega_0^2 = 2\epsilon\delta\lambda \Rightarrow \lambda = \epsilon\delta \pm i\sqrt{\omega_0^2 - \epsilon^2\delta^2} \approx \epsilon\delta \pm i\omega_0$$

Conclusion:

1/ Hopf bifurcation at $\epsilon = 0$

2/ If $\epsilon \ll 1$, slow time-scale on amplification rate and fast time scale on frequency

The Van der Pol Oscillator: a model problem

For $\delta > 0$, at saturation, the instability term $2\epsilon\delta w'$ is cancelled by the nonlinear term $w^2 w'$ when $w^2 \approx 2\epsilon\delta$. We expect that the saturation amplitude is about $\sqrt{\epsilon\delta}$.

We look for the solution under the form:

$$w = \epsilon^{\frac{1}{2}} y$$
$$w_I = \epsilon^{\frac{1}{2}} y_I = \epsilon^{\frac{1}{2}} y(0)$$

Hence:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$
$$y(0) = y_I, y'(0) = 0$$

(Q2a)

The Van der Pol Oscillator: approximations

ODE:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$

Solutions under the form:

$$y = y_0 + \epsilon y_1 + \cdots$$

One time-scale approach:

$$y(t) = y_0(t) + \epsilon y_1(t) + \cdots$$

Two time-scales approach:

$$y = y_0(\mathrm{t}, \tau = \epsilon t) + \epsilon y_1(t, \tau = \epsilon t) + \cdots$$

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The Van der Pol Oscillator: one time-scale approach

ODE:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$

We look for a solution with one time-scale:

$$y = y_0(\mathrm{t}) + \epsilon y_1(t) + o(\epsilon)$$

Then:

$$y' = y_0' + \epsilon y_1' + o(\epsilon)$$
$$y'' = y_0'' + \epsilon y_1'' + o(\epsilon)$$

Initial conditions:

$$y(0) = y_I \Rightarrow y_0(0) + \epsilon y_1(0) + o(\epsilon) = y_I \Rightarrow y_0(0) = y_I, y_1(0) = 0$$
$$y'(0) = 0 \Rightarrow y_0'(0) + \epsilon y_1'(0) + o(\epsilon) = 0 \Rightarrow y_0'(0) = 0, y_1'(0) = 0$$

The Van der Pol Oscillator: one time-scale approach

ODE:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$

$$\Rightarrow y_0'' + \epsilon y_1'' + \omega_0^2(y_0 + \epsilon y_1) = 2\delta\epsilon(y_0' + \epsilon y_1') - \epsilon(y_0' + \epsilon y_1')(y_0 + \epsilon y_1)^2$$

Order ϵ^0:

$$y_0'' + \omega_0^2 y_0 = 0$$

Order ϵ^1:

$$y_1'' + \omega_0^2 y_1 = 2\delta y_0' - y_0^2 y_0'$$

The Van der Pol Oscillator: one time-scale approach

Solution of order ϵ^0, knowing that the solution is real,

$$y_0 = Ae^{i\omega_0 t} + \text{c.c.} = 2|A|\cos(\omega_0 t + \phi)$$

Determination of A with initial conditions:

$$y_0(0) = y_I \Rightarrow 2|A|\cos\phi = y_I$$
$$y_0'(0) = 0 \Rightarrow -2|A|\omega_0 \sin\phi = 0$$

$$\Rightarrow \phi = 0, A = \frac{y_I}{2}$$

The Van der Pol Oscillator: one time-scale approach

We recast this solition into the next order: $y_1'' + \omega_0^2 y_1 = 2\delta y_0' - y_0^2 y_0'$

$$y_1'' + \omega_0^2 y_1 = 2i\omega_0 \delta A e^{i\omega_0 t} + \text{c.c.} - \frac{1}{3}\frac{d\left(A^3 e^{3i\omega_0 t} + 3A^2 A^* e^{i\omega_0 t} + \text{c.c.}\right)}{dt}$$

$$= 2i\omega_0 \delta A e^{i\omega_0 t} + \text{c.c.} - \frac{1}{3}\left(3i\omega_0 A^3 e^{3i\omega_0 t} + 3i\omega_0 A^2 A^* e^{i\omega_0 t} + \text{c.c.}\right)$$

$$= -i\omega_0 A^3 e^{3i\omega_0 t} + i\omega_0 (2\delta A - A^2 A^*) e^{i\omega_0 t} + \text{c.c.}$$

The Van der Pol Oscillator: one time-scale approach

Theorem:

$$y'' + \omega^2 y = \left(a e^{i\Omega t} + \text{c.c.}\right)$$

If $\Omega \neq \omega$, then the most general real solution is (with k as any complex constant):

$$y = \left(k e^{i\omega t} + \frac{a}{\omega^2 - \Omega^2} e^{i\Omega t} + \text{c.c.}\right)$$

If $\Omega = \omega$, then the most general real solution is (with k as any complex constant):

$$y = \left(k e^{i\omega t} - a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c.}\right)$$

Proof (resonant case only):

$$y = k e^{i\omega t} - a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c}$$

$$y' = i\omega k e^{i\omega t} - a\left(\frac{2i\omega}{4\omega^2}\right) e^{i\omega t} - i\omega a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c.}$$

$$y'' = -\omega^2 k e^{i\omega t} + a e^{i\omega t} + a\left(\frac{1 + 2i\omega t}{4}\right) e^{i\omega t} + \text{c.c.}$$

$$y'' + \omega^2 y = a e^{i\omega t} + \text{c.c.}$$

The Van der Pol Oscillator: one time-scale approach

Solution under the form:

$$y_1 = ke^{i\omega_0 t} + \frac{iA^3}{8\omega_0^2}\omega_0 e^{3i\omega_0 t} - (2i\delta A - iA^2A^*)\omega_0\left(\frac{1+2i\omega_0 t}{4\omega_0^2}\right)e^{i\omega_0 t} + \text{c.c.}$$

Determination of k with initial conditions:

$$y_1(0) = 0 \Rightarrow k + \frac{iA^3}{8\omega_0^2}\omega_0 - (2i\delta A - iA^2A^*)\omega_0\left(\frac{1}{4\omega_0^2}\right) + \text{c.c.} \Rightarrow k_r = 0$$

$$y_1'(0) = 0 \Rightarrow ik_i i\omega_0 - \frac{iA^3}{8\omega_0^2}\omega_0 3i\omega_0 - \frac{(2i\delta A - iA^2A^*)}{4\omega_0^2}\omega_0(i\omega_0 + 2i\omega_0) + \text{c.c.} = 0$$

$$k_i = \frac{A^3}{8\omega_0^2}\omega_0 3 + \frac{(2\delta A - A^3)}{8\omega_0^2}\omega_0 6$$

$$\Rightarrow k_i = \frac{-3A^3 + 12\delta A}{8\omega_0}$$

Finally:

$$y_1 = \frac{-3A^3 + 12\delta A}{8\omega_0} ie^{i\omega_0 t} + \frac{iA^3}{8\omega_0}e^{3i\omega_0 t} - (2\delta A - A^3)\left(\frac{1+2i\omega_0 t}{4\omega_0}\right)ie^{i\omega_0 t} + \text{c.c.}$$

The Van der Pol Oscillator: one time-scale approach

Coming back to initial unknown w:

$$w = \epsilon^{\frac{1}{2}} y = \epsilon^{\frac{1}{2}} y_0(\mathrm{t}) + \epsilon^{\frac{3}{2}} y_1(t)$$

$$= \epsilon^{\frac{1}{2}}\left(Ae^{i\omega_0 t} + \text{c.c.}\right)$$

$$+ \epsilon^{\frac{3}{2}}\left(\frac{-3\mathrm{A}^3 + 12\delta\mathrm{A}}{8\omega_0} ie^{i\omega_0 t} + \frac{\mathrm{iA}^3}{8\omega_0}e^{3i\omega_0 t} - (2\delta\mathrm{A} - \mathrm{A}^3)\left(\frac{1+2i\omega_0 t}{4\omega_0}\right)ie^{i\omega_0 t} + \text{c.c.}\right)$$

$$= \left(\tilde{A}e^{i\omega_0 t} + \text{c.c.}\right)$$

$$+ \left(\frac{-3\tilde{A}^3 + 12\tilde{\delta}\tilde{A}}{8\omega_0} ie^{i\omega_0 t} + \frac{\mathrm{i}\tilde{A}^3}{8\omega_0}e^{3i\omega_0 t} - (2\tilde{\delta}\tilde{A} - \tilde{A}^3)\left(\frac{1+2i\omega_0 t}{4\omega_0}\right)ie^{i\omega_0 t} + \text{c.c.}\right)$$

$$\tilde{A} = \frac{w_I}{2}$$

(Q2b)

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The Van der Pol Oscillator: two time-scales approach

PDE:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$

We look for a solution with two time-scales:

$$y = y_0(\mathrm{t}, \tau = \epsilon t) + \epsilon y_1(t, \tau = \epsilon t) + o(\epsilon)$$

Then:

$$y' = \frac{\partial y_0}{\partial t} + \epsilon\left(\frac{\partial y_1}{\partial t} + \frac{\partial y_0}{\partial \tau}\right) + o(\epsilon)$$

$$y'' = \frac{\partial^2 y_0}{\partial t^2} + \epsilon\left(\frac{\partial^2 y_1}{\partial t^2} + 2\frac{\partial^2 y_0}{\partial t \partial \tau}\right) + o(\epsilon)$$

The Van der Pol Oscillator: two time-scales approach

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$

$$\frac{\partial^2 y_0}{\partial t^2} + \epsilon\left(2\frac{\partial^2 y_0}{\partial t \partial \tau} + \frac{\partial^2 y_1}{\partial t^2}\right) + \omega_0^2(y_0 + \epsilon y_1)$$
$$= 2\delta\epsilon\left(\frac{\partial y_0}{\partial t} + \epsilon\left(\frac{\partial y_0}{\partial \tau} + \frac{\partial y_1}{\partial t}\right)\right) - \epsilon\left(\frac{\partial y_0}{\partial t} + \epsilon\left(\frac{\partial y_0}{\partial \tau} + \frac{\partial y_1}{\partial t}\right)\right)(y_0 + \epsilon y_1)^2$$

Order ϵ^0:

$$\frac{\partial^2 y_0}{\partial t^2} + \omega_0^2 y_0 = 0$$

Order ϵ^1:

$$\frac{\partial^2 y_1}{\partial t^2} + \omega_0^2 y_1 = 2\delta\frac{\partial y_0}{\partial t} - y_0^2\frac{\partial y_0}{\partial t} - 2\frac{\partial^2 y_0}{\partial t \partial \tau}$$

The Van der Pol Oscillator: two time-scales approach

Solution at order ϵ^0, knowing that the solution is real,

$$y_0 = A(\tau)e^{i\omega_0 t} + \text{c.c.}$$

We recast this solution into the next order:

$$\frac{\partial^2 y_1}{\partial t^2} + \omega_0^2 y_1$$
$$= 2i\omega_0\delta A e^{i\omega_0 t} + \text{c.c.} - \frac{1}{3}\frac{\partial\left(A^3 e^{3i\omega_0 t} + 3A^2A^* e^{i\omega_0 t} + \text{c.c.}\right)}{\partial t} - 2i\omega_0\frac{dA}{d\tau}e^{i\omega_0 t} + \text{c.c.}$$
$$= 2i\omega_0\delta A e^{i\omega_0 t} + \text{c.c.} - \frac{1}{3}\left(3i\omega_0 A^3 e^{3i\omega_0 t} + 3i\omega_0 A^2A^* e^{i\omega_0 t} + \text{c.c.}\right) - 2i\omega_0\frac{dA}{d\tau}e^{i\omega_0 t} + \text{c.c.}$$
$$= -i\omega_0 A^3 e^{3i\omega_0 t} + i\omega_0\left(2\delta \mathrm{A} - \mathrm{A}^2\mathrm{A}^* - 2\frac{dA}{d\tau}\right)\mathrm{e}^{\mathrm{i}\omega_0 \mathrm{t}} + \text{c.c.}$$

The Van der Pol Oscillator: two time-scales approach

Theorem:

$$y'' + \omega^2 y = \left(a e^{i\Omega t} + \text{c.c.}\right)$$

If $\Omega \neq \omega$, then the most general real solution is (with k as any complex constant):

$$y = \left(k e^{i\omega t} + \frac{a}{\omega^2 - \Omega^2} e^{i\Omega t} + \text{c.c.}\right)$$

If $\Omega = \omega$, then the most general real solution is (with k as any complex constant):

$$y = \left(k e^{i\omega t} - a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c.}\right)$$

Proof (resonant case):

$$y = k e^{i\omega t} - a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c}$$

$$y' = i\omega k e^{i\omega t} - a\left(\frac{2i\omega}{4\omega^2}\right) e^{i\omega t} - i\omega a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c.}$$

$$y'' = -\omega^2 k e^{i\omega t} + a e^{i\omega t} + a\left(\frac{1 + 2i\omega t}{4}\right) e^{i\omega t} + \text{c.c.}$$

$$y'' + \omega^2 y = a e^{i\omega t} + \text{c.c}$$

The Van der Pol Oscillator: two time-scales approach

For the solution to be valid uniformly in time, we kill the secular term:

$$\frac{dA}{d\tau} = \delta \mathrm{A} - \frac{1}{2}\mathrm{A}^2\mathrm{A}^*$$

Final first order solution:

$$w(t) = \epsilon^{\frac{1}{2}}\left(A e^{i\omega_0 t} + \text{c.c}\right) = \left(\tilde{A} e^{i\omega_0 t} + \text{c.c}\right)$$

with:

$$\frac{d\tilde{A}}{dt} = \tilde{\delta}\tilde{A} - \frac{1}{2}\tilde{A}^3$$

$$\tilde{A}(0) = \frac{w_I}{2}$$

(Q2c)

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The Van der Pol Oscillator with forcing term

PDE:

$$w'' + \omega_0^2 w = 2\underbrace{\tilde{\delta}}_{\epsilon\delta} w' - w^2 w' + \underbrace{\tilde{E}}_{\epsilon^{\frac{1}{2}}E} \cos\omega_f t, \epsilon \ll 1, \delta = O(1)$$

We look for the solution under the form:

$$w(t) = \epsilon^{\frac{1}{2}} y(t)$$

Hence:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2 + E\cos\omega_f t$$

The Van der Pol oscillator with forcing term

PDE:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2 + E\cos\omega_f t$$

Expansion:

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + o(\epsilon)$$

Order ϵ^0:

$$\frac{\partial^2 y_0}{\partial t^2} + \omega_0^2 y_0 = E\cos\omega_f t \Rightarrow \mathrm{y}_0 = A(\tau)e^{i\omega_0 t} + \text{c.c.} + 2\underbrace{\frac{1}{2}\frac{E}{\omega_0^2-\omega_f^2}}_{\eta}\cos\omega_f t$$

Order ϵ:

$$\frac{\partial^2 y_1}{\partial t^2} + \omega_0^2 y_1 = 2\delta\frac{\partial y_0}{\partial t} - \frac{1}{3}\frac{\partial y_0^3}{\partial t} - 2\frac{\partial^2 y_0}{\partial t\partial\tau} = i\omega_0 e^{i\omega_0 t}\left(2\delta A - A|A|^2 - 2A\eta^2 - 2\frac{\partial A}{\partial \tau}\right) + \cdots$$

$$y_0^3 = \left(Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t} + \eta e^{i\omega_f t} + \eta e^{-i\omega_f t}\right)^2\left(Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t} + \eta e^{i\omega_f t} + \eta e^{-i\omega_f t}\right)$$
$$= \Big(A^2 e^{2i\omega_0 t} + A^{*2}e^{-2i\omega_0 t} + \eta^2 e^{2i\omega_f t} + \eta^2\ e^{-2i\omega_f t} + 2|A|^2 + 2\eta^2 + 2A\eta e^{i(\omega_0+\omega_f)t}$$
$$+ 2A^*\eta e^{-i(\omega_0+\omega_f)t} + 2A\eta e^{i(\omega_0-\omega_f)t} + 2A^*\eta e^{-i(\omega_0-\omega_f)t}\Big)^2\left(Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t}\right.$$
$$\left. + \eta e^{i\omega_f t} + \eta e^{-i\omega_f t}\right) = (3A|A|^2 + 6\eta^2 A)e^{i\omega_0 t} + \cdots \qquad \omega_f \neq \left(\frac{1}{3}\omega_0, \omega_0, 3\omega_0\right)$$

The Van der Pol oscillator with forcing term

Kill resonant term to remove secular terms:

$$\frac{dA}{d\tau} = \delta A - \frac{1}{2}A|A|^2 - \frac{1}{4}\left(\frac{1}{\omega_0^2-\omega_f^2}\right)^2 E^2 A$$

Final solution:

$$w(t) = \epsilon^{\frac{1}{2}}\left(Ae^{i\omega_0 t} + \text{c.c.} + \frac{E}{\omega_0^2-\omega_f^2}\cos\omega_f t\right) = 2|\tilde{A}|\cos(\omega_0 t + \phi) + \frac{\tilde{E}}{\omega_0^2-\omega_f^2}\cos\omega_f t$$

with:

$$\frac{d\tilde{A}}{dt} = \tilde{\delta}\tilde{\mathrm{A}} - \frac{1}{2}\tilde{\mathrm{A}}^3 - \frac{1}{4}\left(\frac{1}{\omega_0^2-\omega_f^2}\right)^2 \tilde{E}\tilde{\mathrm{A}}$$

Or:

$$\frac{d\tilde{A}}{dt} = \left[\tilde{\delta} - \frac{1}{4}\left(\frac{\tilde{E}}{\omega_0^2-\omega_f^2}\right)^2\right]\tilde{\mathrm{A}} - \frac{1}{2}\tilde{\mathrm{A}}^3$$

(Q3)

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The Ginzburg-Landau eq.

Forcing with $\omega_f \neq \omega_0$

The forced non-linear Ginzburg-Landau equations read:

$$\partial_t w + \mathcal{L}w + w|w|^2 = f(x,t)$$

$$\mathcal{L} = U\partial_x - \mu(x) - \gamma\partial_{xx}, \qquad \mu(x) = i\omega_0 + \mu_0 - \gamma\chi^4 x^2$$

and $f(x,t)$ is a forcing. The least-damped global mode is $\widehat{w}(x) = \zeta e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$ with eigenvalue $\lambda = \mathrm{i}\omega_0 + \mu_0 - \mu_c$ where $\mu_c = \frac{U^2}{4\gamma} + \gamma\chi^2$. The corresponding adjoint global mode is $\widetilde{w}(x) = \xi e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$. They are normalized such that: $\langle \widehat{w}, \widehat{w} \rangle = \langle \widetilde{w}, \widehat{w} \rangle = 1$.

We choose μ_0 in the vicinity of μ_c such that:

$$\mu_0 = \mu_c + \tilde{\delta},$$

Where $\tilde{\delta} = \epsilon\delta$ with $0 < \epsilon \ll 1$, $\delta = O(1)$.

Hence, the full perturbed operator $\mathcal{L}$ reads:

$$\mathcal{L} = U\partial_x - (i\omega_0 + \mu_c + \epsilon\delta - \gamma\chi^4 x^2) - \gamma\partial_{xx} = \mathcal{L}_c - \epsilon\delta$$

The Ginzburg-Landau eq.

Forcing with $\omega_f \neq \omega_0$

We choose a forcing such that:

$$f(x,t) = \tilde{E}\hat{f}(x)e^{i\omega_f t}$$

where $\tilde{E} = \epsilon^{\frac{1}{2}}E$, $E = O(1)$ is the forcing amplitude (positive real). The forcing frequency ω_f is such that $\omega_f \neq \omega_0$.

The solution is sought under the form:

$$w = \epsilon^{\frac{1}{2}}y$$

What is the equation governing y?

We postulate that:

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \cdots$$

where $\tau = \epsilon t$ is a slow time-scale.
What is the equation governing y_0? What is the equation governing y_1?

The Ginzburg-Landau eq.

Forcing with $\omega_f \neq \omega_0$

$$\partial_t w + \mathcal{L}w + w|w|^2 = f(x,t)$$
$$\Rightarrow \partial_t y + (\mathcal{L}_c - \epsilon\delta)y + \epsilon y|y|^2 = E\hat{f}(x)e^{i\omega_f t}$$

Then:

$$\partial_t y = \partial_t y_0 + \epsilon(\partial_t y_1 + \partial_\tau y_0)$$

Throwing everything inside:

$$\partial_t y_0 + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots + \mathcal{L}_c(y_0 + \epsilon y_1 + \cdots)$$
$$= \epsilon\delta y_0 + \cdots - \epsilon y_0|y_0|^2 + E\hat{f}(x)e^{i\omega_f t}$$

Order 1:

$$\partial_t y_0 + \mathcal{L}_c y_0 = E\hat{f}(x)e^{i\omega_f t}$$

Orderϵ:

$$\partial_t y_1 + \partial_\tau y_0 + \mathcal{L}_c y_1 = \delta y_0 - y_0|y_0|^2$$

The Ginzburg-Landau eq.

Forcing with $\omega_f \neq \omega_0$

Show that $y_0(t,\tau) = A(\tau)e^{i\omega_0 t}\widehat{w}_c + Ee^{i\omega_f t}\widehat{w}_p$ is an acceptable solution for y_0, with $\widehat{w}_p$ a spatial structure to be defined depending on $\hat{f}(x)$ and ω_f.

Show that the solution $y_1(t,\tau)$ is bounded only if:

$$\frac{dA}{d\tau} = (\lambda\delta - \pi|E|^2)A - \mu A|A|^2$$

$$\lambda = 1$$

$$\mu = \langle \widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2 \rangle$$

$$\pi = 2\left\langle \widetilde{w}_c, \widehat{w}_c\left|\widehat{w}_p\right|^2 \right\rangle$$

The Ginzburg-Landau eq.

Forcing with $\omega_f \neq \omega_0$

Order 1:

$$\partial_t y_0 + \mathcal{L}_c y_0 = E\hat{f}(x)e^{i\omega_f t}$$

1/ Homogeneous solution: $A(\tau)e^{i\omega_0 t}\widehat{w}_c$ since

$$i\omega_0\widehat{w}_c + \mathcal{L}_c\widehat{w}_c = 0$$

2/ Particular solution: $Ee^{i\omega_f t}\widehat{w}_p$ with

$$i\omega_f\widehat{w}_p + \mathcal{L}_c\widehat{w}_p = \hat{f}(x)$$

3/ General solution:

$$y_0 = A(\tau)e^{i\omega_0 t}\widehat{w}_c + Ee^{i\omega_f t}\widehat{w}_p$$

The Ginzburg-Landau eq.

Forcing with $\omega_f \neq \omega_0$

Order ϵ :

$$\partial_t y_1 + \mathcal{L}_c y_1 = -\frac{dA}{d\tau} e^{i\omega_0 t}\widehat{w}_c + \delta A e^{i\omega_0 t}\widehat{w}_c - A|A|^2 e^{i\omega_0 t}\widehat{w}_c|\widehat{w}_c|^2 - 2A|E|^2 e^{i\omega_0 t}\widehat{w}_c\left|\widehat{w}_p\right|^2 + \cdots$$

$$y_0|y_0|^2 = \left(Ae^{i\omega_0 t}\widehat{w}_c + Ee^{i\omega_f t}\widehat{w}_p\right)\left(Ae^{i\omega_0 t}\widehat{w}_c + Ee^{i\omega_f t}\widehat{w}_p\right)\left(A^* e^{-i\omega_0 t}\widehat{w}_c^* + E^* e^{-i\omega_f t}\widehat{w}_p^*\right)$$
$$= A|A|^2 e^{i\omega_0 t}\widehat{w}_c|\widehat{w}_c|^2 + 2A|E|^2 e^{i\omega_0 t}\widehat{w}_c\left|\widehat{w}_p\right|^2 + \cdots$$

Kill resonant terms:

$$\frac{dA}{d\tau} = \delta A - A|A|^2\langle\widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2\rangle - 2A|E|^2\left\langle\widetilde{w}_c, \widehat{w}_c\left|\widehat{w}_p\right|^2\right\rangle$$

Theorem: Let w be the solution of the following equation

$$\partial_t w + \mathcal{L}\mathrm{w} = e^{i\omega t} f$$

1/ If $i\omega$ does not belong to the eigenvalues of the Jacobian, then the most general real solution to this equation is

$$w = \sum k_j e^{\lambda_j t}\widehat{w}_j + e^{i\omega t}\widehat{w}_p$$

With the particular solution determined from:

$$i\omega\widehat{w}_p + \mathcal{L}\widehat{w}_p = f$$

And the homogeneous one from the global modes:

$$\lambda_j\widehat{w}_j + \mathcal{L}\widehat{w}_j = 0$$

2/ If $(\mathrm{i}\omega, \widehat{w})$ corresponds to one of the eigenvalues of the Jacobian:

$$\mathrm{i}\omega\widehat{w} + \mathcal{L}\widehat{w} = 0$$

Then, the most general real solution is

$$w(t) = \sum k_j e^{\lambda_j t}\widehat{w}_j + e^{i\omega t} < \widetilde{w}, f > t\widehat{w} + \sum_{\omega_j \neq \omega} \frac{< \widetilde{\mathrm{w}}_j, f >}{i\omega - \lambda_j} e^{i\omega t}\widehat{w}_j$$

Where $\widetilde{w}_j$ (including $\widetilde{w}$) is the bi-orthogonal basis corresponding to $\widehat{w}_k$ (including $\widehat{w}$)

$$< \widetilde{w}_j, \widehat{w}_k > = \delta_{jk}$$

Proof: particular solution with method of variation of constant

$$w(t) = \alpha(t)e^{i\omega t}\widehat{w} + \sum_{\omega_j \neq \omega} \alpha_j e^{i\omega t}\widehat{w}_j$$

Inserting this solution in governing equation:

$$\frac{d\alpha}{dt}\widehat{w} + \alpha(\mathrm{i}\omega\widehat{w} + \mathcal{L}\widehat{w}) + \sum_{\omega_j \neq \omega} \alpha_j\left(i\omega\widehat{w}_j + \mathcal{L}\widehat{w}_j\right) = f$$

$$\frac{d\alpha}{dt}\widehat{w} + \sum_{\omega_j \neq \omega} \alpha_j\left(i\omega - \lambda_j\right)\widehat{w}_j = f$$

Scalar product with $\widetilde{w}$ and $\widetilde{w}_j \neq \widetilde{w}$:

$$\frac{d\alpha}{dt} = <\widetilde{w}, f>$$

$$\alpha_j = \frac{\langle \widetilde{w}_j, f\rangle}{i\omega - \lambda_j}$$

The Ginzburg-Landau eq.

Forcing with $\omega_f \neq \omega_0$

Show that the leading–order solution of the flowfield may be given by:

$$w(x,t) = \tilde{A}(t)e^{i\omega_0 t}\widehat{w}_c(x) + \tilde{E}e^{i\omega_f t}\widehat{w}_p(x)$$

where:

$$\frac{d\tilde{A}}{dt} = \left(\lambda\tilde{\delta} - \pi\left|\tilde{E}\right|^2\right)\tilde{A} - \mu\tilde{A}\left|\tilde{A}\right|^2$$

Hint: note that $\tilde{A} = \epsilon^{\frac{1}{2}}A(\tau)$

Discussion

Amplitude equation:

$$\frac{d\tilde{A}}{dt} = \left(\lambda\tilde{\delta} - \pi\left|\tilde{E}\right|^2\right)\tilde{A} - \mu\tilde{A}\left|\tilde{A}\right|^2$$

Polar coordinates:

$$\tilde{A} = re^{i\phi}$$

Inserting this expression into the amplitude equation:

$$\frac{dr}{dt}e^{i\phi} + ri\frac{d\phi}{dt}e^{i\phi} = \left((\lambda_r + i\lambda_i)\tilde{\delta} - (\pi_r + i\pi_i)\left|\tilde{E}\right|^2\right)re^{i\phi} - (\mu_r + i\mu_i)r^3e^{i\phi}$$

Hence, removing $e^{i\phi}$ and taking the real and imaginary parts:

$$\frac{d}{dt}(\ln r) = \lambda_r\tilde{\delta} - \pi_r\left|\tilde{E}\right|^2 - \mu_r r^2$$

$$\frac{d\phi}{dt} = \lambda_i\tilde{\delta} - \pi_i\left|\tilde{E}\right|^2 - \mu_i r^2$$

Discussion with $\left|\tilde{A}\right| = O(1)$ and $\tilde{E} = 0$

Amplitude equations:

$$\frac{d}{dt}(\ln r) = \lambda_r\tilde{\delta} - \mu_r r^2$$

$$\frac{d\phi}{dt} = \lambda_i\tilde{\delta} - \mu_i r^2$$

Fixed point for r if $\mu_r > 0$:

$$\frac{d}{dt}(\ln r) = 0 \Rightarrow r = \sqrt{\frac{\lambda_r}{\mu_r}\tilde{\delta}}$$

Square root of $\tilde{\delta}$!

Frequency shift on limit-cycle: $\frac{d\phi}{dt} = \left(\lambda_i - \mu_i\frac{\lambda_r}{\mu_r}\right)\tilde{\delta}$

Solution on limit-cycle: $w = w_0 + \sqrt{\frac{\lambda_r}{\mu_r}\tilde{\delta}}\left[e^{i\left(\omega_0 + \left(\lambda_i - \mu_i\frac{\lambda_r}{\mu_r}\right)\tilde{\delta}\right)t}y_A + \text{c.c}\right] + \cdots$

Actual frequency on limit-cycle: $\omega = \omega_0 + \underbrace{\lambda_i\tilde{\delta}}_{\text{Linear}} \underbrace{-\mu_i\frac{\lambda_r}{\mu_r}\tilde{\delta}}_{\text{Non-linear interaction}}$

Discussion with $|\tilde{A}| \ll 1$ and $\tilde{E} > 0$

Amplitude equation:

$$\frac{d}{dt}(\ln r) = \lambda_r \tilde{\delta} - \pi_r |\tilde{E}|^2$$
$$\frac{d\phi}{dt} = \lambda_i \tilde{\delta} - \pi_i |\tilde{E}|^2$$

The solution is:

$$w = w_0 + \tilde{A}_0 e^{[\lambda_r \tilde{\delta} - \pi_r |\tilde{E}|^2 + i(\omega_0 + \lambda_i \tilde{\delta} - \pi_i |\tilde{E}|^2)]t} y_A + \text{c.c}$$

The eigenvalue is:

$$\lambda_r \tilde{\delta} - \pi_r |\tilde{E}|^2 + i\left(\omega_0 + \lambda_i \tilde{\delta} - \pi_i |\tilde{E}|^2\right)$$

Amplification rate:

$$\sigma = \lambda_r \tilde{\delta} - \pi_r |\tilde{E}|^2$$

If $\pi_r > 0$, linearly stable if:

$$\lambda_r \tilde{\delta} - \pi_r |\tilde{E}|^2 < 0 \Rightarrow |\tilde{E}| > \sqrt{\frac{\lambda_r}{\pi_r}\tilde{\delta}}$$

Outline

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- The Van der Pol oscillator: a model problem
  - ➢ One time-scale approach
  - ➢ Two times-scales approach
  - ➢ With forcing term
- The Ginzburg-Landau eq.
  - ➢ Forcing with $\omega_f \neq \omega_0$
  - ➢ Forcing with perturbation of linear operator ($\omega_f = 0$)
  - ➢ Forcing with ω_f close to ω_0
- The forced Navier-Stokes eq. with cylinder flow
  - ➢ Forcing with ω_f close to 0
  - ➢ Forcing with ω_f close to ω_0
  - ➢ Forcing with $\omega_f \neq \omega_0$
- Illustration for forced Navier-Stokes eq. with open-cavity flow

The Ginzburg-Landau eq.

Forcing with perturbation of linear operator

The forced non-linear Ginzburg-Landau equations read:

$$\partial_t w + \mathcal{L}w + w|w|^2 = 0$$

$$\mathcal{L} = U\partial_x - \mu(x) - \gamma\partial_{xx}, \qquad \mu(x) = i\omega_0 + \mu_0 - \gamma\chi^4 x^2$$

and $f(x,t)$ is a forcing. The least-damped global mode is $\hat{w}(x) = \zeta e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$ with eigenvalue $\lambda = \mathrm{i}\omega_0 + \mu_0 - \mu_c$ where $\mu_c = \frac{U^2}{4\gamma} + \gamma\chi^2$. The corresponding adjoint global mode is $\tilde{w}(x) = \xi e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$. They are normalized such that: $\langle \hat{w}, \hat{w} \rangle = \langle \tilde{w}, \hat{w} \rangle = 1$.

We choose μ_0 in the vicinity of μ_c such that:

$$\mu_0 = \mu_c + \tilde{\delta},$$

Where $\tilde{\delta} = \epsilon\delta$ with $0 < \epsilon \ll 1$, $\delta = O(1)$.

We consider that the operator $\mathcal{L}$ may be perturbed by an arbitrary perturbation operator (which will be defined later):

$$\widetilde{\Delta\mathcal{L}} = \tilde{E}\Delta\mathcal{L} = \epsilon E\Delta\mathcal{L}.$$

Hence, the full perturbed operator $\mathcal{L}$ reads:

$$\mathcal{L} = U\partial_x - (i\omega_0 + \mu_c + \epsilon\delta - \gamma\chi^4 x^2) - \gamma\partial_{xx} + \epsilon E\Delta\mathcal{L} = \mathcal{L}_c - \epsilon\delta + \epsilon E\Delta\mathcal{L}$$

The Ginzburg-Landau eq.

Forcing with perturbation of linear operator

The solution is sought under the form:

$$w = \epsilon^{\frac{1}{2}} y$$

What is the equation governing y?

We postulate that:

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \cdots$$

where $\tau = \epsilon t$ is a slow time-scale.
What is the equation governing y_0? What is the equation governing y_1?

The Ginzburg-Landau eq.
Forcing with perturbation of linear operator

$$\partial_t w + \mathcal{L}w + w|w|^2 = 0$$
$$\Rightarrow \partial_t y + (\mathcal{L}_c - \epsilon\delta + \epsilon E\Delta\mathcal{L})y + \epsilon y|y|^2 = 0$$

Then:

$$\partial_t y = \partial_t y_0 + \epsilon(\partial_t y_1 + \partial_\tau y_0)$$

Throwing everything inside:

$$\partial_t y_0 + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots + \mathcal{L}_c(y_0 + \epsilon y_1 + \cdots) = \epsilon\delta y_0 - \epsilon E\Delta\mathcal{L}y_0 + \cdots - \epsilon y_0|y_0|^2$$

Order 1:

$$\partial_t y_0 + \mathcal{L}_c y_0 = 0$$

Orderϵ:

$$\partial_t y_1 + \partial_\tau y_0 + \mathcal{L}_c y_1 = \delta y_0 - E\Delta\mathcal{L}y_0 - y_0|y_0|^2$$

The Ginzburg-Landau eq.
Forcing with perturbation of linear operator

Show that $y_0(t,\tau) = A(\tau)e^{i\omega_0 t}\hat{w}_c$ is an acceptable solution for y_0.

Show that the solution $y_1(t,\tau)$ is bounded only if:

$$\frac{dA}{d\tau} = (\lambda\delta + \pi E)A - \mu A|A|^2$$
$$\lambda = 1$$
$$\mu = \langle\tilde{w}_c, \hat{w}_c|\hat{w}_c|^2\rangle$$
$$\pi = -\langle\tilde{w}_c, \Delta\mathcal{L}\hat{w}_c\rangle$$

The Ginzburg-Landau eq.
Forcing with perturbation of linear operator

Order 1:

$$\partial_t y_0 + \mathcal{L}_c y_0 = 0$$

1/ Homogeneous solution: $A(\tau)e^{i\omega_0 t}\widehat{w}_c$ since

$$i\omega_0\widehat{w}_c + \mathcal{L}_c\widehat{w}_c = 0$$

2/ General solution:

$$y_0 = A(\tau)e^{i\omega_0 t}\widehat{w}_c$$

The Ginzburg-Landau eq.
Forcing with perturbation of linear operator

Order ϵ :

$$\partial_t y_1 + \mathcal{L}_c y_1 = -\frac{dA}{d\tau}e^{i\omega_0 t}\widehat{w}_c + \delta A e^{i\omega_0 t}\widehat{w}_c - AEe^{i\omega_0 t}\Delta\mathcal{L}\widehat{w}_c - A|A|^2 e^{i\omega_0 t}\widehat{w}_c|\widehat{w}_c|^2 \cdots$$

$$y_0|y_0|^2 = A|A|^2 e^{i\omega_0 t}\widehat{w}_c|\widehat{w}_c|^2 + \cdots$$

Kill resonant terms:

$$\frac{dA}{d\tau} = \delta A - AE\langle\widetilde{w}_c, \Delta\mathcal{L}\widehat{w}_c\rangle - A|A|^2\langle\widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2\rangle$$

The Ginzburg-Landau eq.

Forcing with perturbation of linear operator

Show that the leading–order solution of the flowfield may be given by:

$$w(x,t) = \tilde{A}(t)e^{i\omega_0 t}\hat{w}_c(x)$$

where:

$$\frac{d\tilde{A}}{dt} = (\lambda\tilde{\delta} + \pi\tilde{E})\tilde{A} - \mu\tilde{A}|\tilde{A}|^2$$

$$\pi = -\langle\tilde{w}_c, \Delta\mathcal{L}\hat{w}_c\rangle$$

Hint: note that $\tilde{A} = \epsilon^{\frac{1}{2}}A(\tau)$

For $|\tilde{A}| \ll 1$: The solution is:

$$w = w_0 + \tilde{A}_0 e^{[i\omega_0 + \lambda\tilde{\delta} + \pi\tilde{E}]t} y_A + \text{c.c}$$

New eigenvalue is $i\omega_0 + \lambda\tilde{\delta} + \pi\tilde{E}$

Stabilization of flow for (if $\pi_r > 0$):

$$\tilde{E} > \frac{\lambda_r}{\pi_r}\tilde{\delta}$$

Frequency shift:

$$\lambda_i\tilde{\delta} + \pi_i\tilde{E}$$

Open-loop control

$|\tilde{A}| \ll 1$

Open-loop control ($\tilde{A} \ll 1$) that modifies the stability characteristics of the flow:

$$\mathcal{L}_c = \text{U}\partial_x - \mu_c(x) - \gamma\partial_{xx}$$

Where:

$$\mu_c(x) = i\omega_0 + \mu_c - \gamma\chi^4 x^2$$

If:

$$\mu_c(x) := \mu_c(x) + \tilde{E}\delta\mu(x)$$

Then:

$$\mathcal{L}_c \coloneqq \mathcal{L}_c - \tilde{E}\delta\mu(x) = \mathcal{L}_c + \tilde{E}\Delta\mathcal{L}$$

So that:

$$\Delta\mathcal{L} = -\delta\mu$$

Where should $\delta\mu(x)$ be located to achieve maximal stabilization?

Open-loop control
$|\tilde{A}| \ll 1$

The new eigenvalue is:

$$i\omega_0 + \lambda\tilde{\delta} + \pi\tilde{E}$$

$$\pi = -\langle \widetilde{w}_c, \Delta\mathcal{L}\widehat{w}_c \rangle = -\int_{-\infty}^{+\infty} \widetilde{w}_c(x)\delta\mu(x)\widehat{w}_c(x)dx$$

$\delta\mu$ should be located where $\widetilde{w}_c(x)\widehat{w}_c(x)$ is strong => overlap region of $\widetilde{w}_c(x)$ and $\widehat{w}_c(x)$

Closed-loop control
$|\tilde{A}| \ll 1$

Closed-loop control ($|\tilde{A}| \ll 1$) that modifies the stability characteristics of the flow:

$$\mathcal{L}_c \coloneqq \mathcal{L}_c + \tilde{E}\Delta\mathcal{L}w = \mathcal{L}_c + \tilde{E}\delta(x - x_a)w(x_s)$$

=> we actuate at $x = x_a$ with a control law $u = \tilde{E}w(x_s)$, ie proportionally to a measure of the state at x_s.

Where should x_a be located to achieve optimal stabilization with minimal control amplitude u?
Where should x_s be located to achieve minimal gain $\tilde{E}$?
What is the asymptotic value of u when the system is stabilized?

Discussion with $\widetilde{\Delta\mathcal{L}} \neq 0$ (closed-loop) $|\tilde{A}| \ll 1$

The new eigenvalue is:

$$i\omega_0 + \lambda\tilde{\delta} + \pi\tilde{E}$$

$$\pi = -\langle \widetilde{w}_c, \Delta\mathcal{L}\widehat{w}_c \rangle = -\int_{-\infty}^{+\infty} \widetilde{w}_c(x)\delta(x - x_a)\widehat{w}_c(x_s)dx = -\widetilde{w}_c(x_a)\,\widehat{w}_c(x_c)$$

Actuator should be located at maximal amplitude of adjoint.
Sensor should be located at maximal amplitude of direct mode.

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The Ginzburg-Landau eq.

Forcing with $\omega_f \approx \omega_0$

The forcing frequency ω_f is chosen in the vicinity of the natural frequency ω_0 of the flow:

$$\omega_f = \omega_0 + \tilde{\Omega}$$

where $\tilde{\Omega} = \epsilon\Omega$, $\epsilon \ll 1, \Omega = O(1)$.
The forcing amplitude is:

$$\tilde{E} = \epsilon^{\frac{3}{2}} E, E = O(1)$$

The solution is sought under the form:

$$w = \epsilon^{\frac{1}{2}} y$$

What is the equation governing y?
We look for a solution under the form:

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \cdots$$

where $\tau = \epsilon t$ is a slow time-scale.
What is the equation governing y_0? What is the equation governing y_1?
Show that $y_0(t,\tau) = A(\tau) e^{i\omega_0 t} \hat{w}_c(x)$ is an acceptable solution for y_0.
Show that the solution $y_1(t,\tau)$ is bounded only if:

$$\frac{dA}{d\tau} = \lambda\delta A - \mu A|A|^2 + \pi E e^{\mathrm{i}\Omega\tau}$$

$$\lambda = 1, \mu = <\tilde{w}_c, \hat{w}_c|\hat{w}_c|^2>, \pi = \overline{\tilde{w}_c(x_f)}$$

The Ginzburg-Landau eq.

Forcing with $\omega_f \approx \omega_0$

$$\partial_t y + \mathcal{L}_c y = \epsilon\delta y - \epsilon y|y|^2 + \epsilon E\delta(x - x_f) e^{i\omega_0 t} e^{\mathrm{i}\Omega\tau}$$

$$\partial_t y = \partial_t y_0 + \epsilon(\partial_t y_1 + \partial_\tau y_0)$$

Throwing everything inside:

$$\partial_t y_0 + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \mathcal{L}_c(y_0 + \epsilon y_1) = \epsilon\delta y_0 - \epsilon y_0|y_0|^2 + \epsilon E\delta(x - x_f) e^{i\omega_0 t} e^{\mathrm{i}\Omega\tau}$$

Order 1:

$$\partial_t y_0 + \mathcal{L}_c y_0 = 0$$
$$y_0 = A(\tau) e^{i\omega_0 t} \hat{w}_c$$

Order ϵ:

$$\partial_t y_1 + \partial_\tau y_0 + \mathcal{L}_c y_1 = \delta y_0 - y_0|y_0|^2 + E\delta(x - x_f) e^{i\omega_0 t} e^{\mathrm{i}\Omega\tau}$$

The Ginzburg-Landau eq.

Forcing with $\omega_f \approx \omega_0$

$$\partial_t y_1 + \mathcal{L}_c y_1 = -\frac{dA}{d\tau} e^{i\omega_0 t}\widehat{w}_c + \delta A e^{i\omega_0 t}\widehat{w}_c - A|A|^2 e^{i\omega_0 t}\widehat{w}_c|\widehat{w}_c|^2 + E\delta(x - x_f)e^{i\omega_0 t}e^{i\Omega\tau}$$

Compatibility condition:

$$< \widetilde{w}_c, -\frac{dA}{d\tau} e^{i\omega_0 t}\widehat{w}_c + \delta A e^{i\omega_0 t}\widehat{w}_c - A|A|^2 e^{i\omega_0 t}\widehat{w}_c|\widehat{w}_c|^2 + E\delta(x - x_f)e^{i\omega_0 t}e^{i\Omega\tau} >= 0$$

$$< \widetilde{w}_c, -\frac{dA}{d\tau}\phi + \delta A\widehat{w}_c - A|A|^2\widehat{w}_c|\widehat{w}_c|^2 + E\delta(x - x_f)e^{i\Omega\tau} >= 0$$

$$-\frac{dA}{d\tau} < \widetilde{w}_c, \widehat{w}_c > +\delta A < \widetilde{w}_c, \widehat{w}_c > -A|A|^2 < \widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2 > +E e^{i\Omega\tau} < \widetilde{w}_c, \delta(x - x_f) > = 0$$

$$\frac{dA}{d\tau} = \delta A - < \widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2 > A|A|^2 + \overline{\widetilde{w}_c(x_f)} E e^{i\Omega\tau}$$

The Ginzburg-Landau eq.

Forcing with $\omega_f \approx \omega_0$

Show that the leading-order solution of the flowfield may be given by:

$$w(x,t) = \tilde{C}(t)e^{i\omega_f t}\widehat{w}_c(x)$$

where:

$$\frac{d\tilde{C}}{dt} = (-i\tilde{\Omega} + \lambda\tilde{\delta})\tilde{C} - \mu\tilde{C}|\tilde{C}|^2 + \pi\tilde{E}$$

Hint: note that $\tilde{C}$ verifies $\tilde{C} = \epsilon^{\frac{1}{2}}A(\tau)e^{-i\Omega\tau}$

Numerical simulations of the equation governing C' show that there exists a threshold amplitude $\tilde{E}_{\text{LOCK}}$, such that: If $\tilde{E} > \tilde{E}_{\text{LOCK}}$, then $\tilde{C} \to \tilde{C}_{LOCK}$ as $t \to \infty$, where $\tilde{C}_{LOCK}$ is a complex constant. What is the frequency of the flowfield in this case? Can you comment this result in terms of open-loop control? How should the forcing location x_f be chosen to minimize the threshold amplitude $\tilde{E}_{\text{LOCK}}$?

The Ginzburg-Landau eq.

Forcing with $\omega_f \approx \omega_0$

$$\frac{d\tilde{C}}{dt} = \epsilon^{\frac{1}{2}}\left(-i\Omega + \frac{1}{A}\frac{dA}{d\tau}\right)Ae^{-i\Omega\tau}\epsilon = \epsilon^{\frac{1}{2}}\left(-i\Omega + \lambda\delta - \mu|A|^2 + \pi\frac{E}{A}e^{i\Omega\tau}\right)Ae^{-i\Omega\tau}\epsilon$$

$$= \left(-i\tilde{\Omega} + \lambda\tilde{\delta} - \mu\left|\tilde{C}\right|^2 + \epsilon\pi\frac{E}{\tilde{C}\epsilon^{-\frac{1}{2}}e^{i\Omega\tau}}e^{i\Omega\tau}\right)\tilde{C}$$

$$\frac{d\tilde{C}}{dt} = -i\tilde{\Omega}\tilde{C} + \lambda\tilde{\delta}\tilde{C} - \mu\tilde{C}\left|\tilde{C}\right|^2 + \pi\tilde{E}$$

$$w = \tilde{C}(t)e^{i\omega_f t}\hat{w}_c$$

Locking!

The Ginzburg-Landau eq.

Forcing with $\omega_f \approx \omega_0$

Show that the leading-order solution of the flowfield may be given by:

$$w(x,t) = \tilde{D}(t)\hat{w}_c(x)$$

where:

$$\frac{d\tilde{D}}{dt} = (i\omega_0 + \lambda\tilde{\delta})\tilde{D} - \mu\tilde{D}\left|\tilde{D}\right|^2 + \pi\tilde{E}e^{i\omega_f t}$$

Hint: note that $\tilde{D}$ verifies $\tilde{D} = \epsilon^{\frac{1}{2}}A(\tau)e^{i\omega_0 t}$

What is the transfer function of the flow?

Where is the optimal place to locate the actuator?

The Ginzburg-Landau eq.

Forcing with $\omega_f \approx \omega_0$

$$\frac{d\tilde{D}}{dt} = \epsilon^{\frac{1}{2}}\left(i\omega_0 + \epsilon\frac{1}{A}\frac{dA}{d\tau}\right)Ae^{i\omega_0 t} = \epsilon^{\frac{1}{2}}\left(i\omega_0 + \lambda\delta - \mu|A|^2 + \pi\frac{E}{A}e^{i\Omega\tau}\right)Ae^{i\omega_0 t}$$

$$= \left(i\omega_0 + \lambda\tilde{\delta} - \mu\left|\tilde{D}\right|^2 + \epsilon\pi\frac{E}{\tilde{D}\epsilon^{-\frac{1}{2}}e^{-i\omega_0 t}}e^{i\Omega\tau}\right)\tilde{D}$$

$$\frac{d\tilde{D}}{dt} = i\omega_0\tilde{D} + \lambda\tilde{\delta}\tilde{D} - \mu\tilde{D}\left|\tilde{D}\right|^2 + \pi\tilde{E}e^{i\omega_f t}$$

$$w = \tilde{D}(t)\,\hat{w}_c$$

For $\tilde{\delta} < 0$ (stable flow):

$$\tilde{D} = \hat{D}e^{i\omega_f t}$$

$$\frac{\hat{D}}{\tilde{E}} = \frac{\pi}{i(\omega_f - \omega_0) - \lambda\tilde{\delta} + \mu\left|\hat{D}\right|^2}$$

$$\pi = \overline{\tilde{w}_c(x_f)}$$

Maximal effect if x_f is close to maximum of adjoint. Response at

Outline

- Vortex shedding in cylinder flow at $Re = 100$
- The Van der Pol oscillator: a model problem
  - One time-scale approach
  - Two times-scales approach
  - With forcing term
- The Ginzburg-Landau eq.
  - Forcing with $\omega_f \neq \omega_0$
  - Forcing with perturbation of linear operator ($\omega_f = 0$)
  - Forcing with ω_f close to ω_0
- The forced Navier-Stokes eq. with cylinder flow
  - Forcing with ω_f close to 0
  - Forcing with ω_f close to ω_0
  - Forcing with $\omega_f \neq \omega_0$
- Illustration for forced Navier-Stokes eq. with open-cavity flow

The forced Navier-Stokes equations

Forced Navier-Stokes equations with viscosity $\nu = \nu_c - \tilde{\delta}$:

$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = \tilde{\delta}\mathcal{M}w + \tilde{E}f$$

$$w = \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} -\nu_c \Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}, \mathcal{M} = \begin{pmatrix} -\Delta & 0 \\ 0 & 0 \end{pmatrix}$$

Scalings:

$$\tilde{\delta} = \epsilon\delta, \qquad \delta = O(1)$$
$$w = w_0 + \epsilon^{1/2} y, \qquad \epsilon \ll 1,$$
$$\tilde{E} = \epsilon E, \qquad E = O(1)$$

Base-flow:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0$$

Perturbation dynamics:

$$\mathcal{B}\partial_t y + \mathcal{N}_{w_0} y + \mathcal{L}y = \epsilon^{\frac{1}{2}}\delta\mathcal{M}w_0 + \epsilon\delta\mathcal{M}y - \frac{1}{2}\epsilon^{\frac{1}{2}}\mathcal{N}(y,y) + \epsilon^{\frac{1}{2}}Ef$$

The forced Navier-Stokes equations

Expansion:

$$y = y_0(t, \tau = \epsilon t) + \epsilon^{1/2} y_{1/2}(t, \tau = \epsilon t) + \epsilon^1 y_1(t, \tau = \epsilon t) + \cdots$$

In particular:

$$\partial_t y = \partial_t y_0 + \epsilon^{1/2}\partial_t y_{1/2} + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots$$

The forced Navier-Stokes equations

$$\mathcal{B}\left(\partial_t y_0 + \epsilon^{1/2}\partial_t y_{1/2} + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots\right) + (\mathcal{N}_{w_0} + \mathcal{L})(y_0 + \epsilon^{1/2} y_{1/2} + \epsilon^1 y_1)$$
$$= \epsilon^{\frac{1}{2}}\delta\mathcal{M}w_0 + \epsilon\delta\mathcal{M}y_0 - \frac{1}{2}\epsilon^{\frac{1}{2}}\mathcal{N}(y_0, y_0) - \epsilon\mathcal{N}\left(y_0, y_{1/2}\right) + \epsilon^{\frac{1}{2}}Ef$$

$$\Rightarrow$$

$$\mathcal{B}\partial_t y_0 + \mathcal{N}_{w_0} y_0 + \mathcal{L}y_0 = 0$$
$$\mathcal{B}\partial_t y_{1/2} + \mathcal{N}_{w_0} y_{1/2} + \mathcal{L}y_{1/2} = \delta\mathcal{M}w_0 - \frac{1}{2}\mathcal{N}(y_0, y_0) + Ef$$
$$\mathcal{B}\partial_t y_1 + \mathcal{N}_{w_0} y_1 + \mathcal{L}y_1 = -\mathcal{B}\partial_\tau y_0 + \delta\mathcal{M}y_0 - \mathcal{N}\left(y_0, y_{\frac{1}{2}}\right)$$

Order ϵ^0

PDE:

$$\mathcal{B}\partial_t y_0 + \mathcal{N}_{w_0} y_0 + \mathcal{L}y_0 = 0$$

We choose:

$$y_0 = \left(A(\tau)e^{i\omega_0 t}y_A + \text{c.c}\right)$$

Order $\epsilon^{\frac{1}{2}}$

PDE:

$$\mathcal{B}\partial_t y_{1/2} + \mathcal{N}_{w_0} y_{1/2} + \mathcal{L} y_{1/2} = \delta \mathcal{M} w_0 - \frac{1}{2}\mathcal{N}(y_0, y_0)$$

$$= \delta \mathcal{M} w_0 + \left(-\frac{1}{2}A^2 e^{2i\omega_0 t}\mathcal{N}(y_A, y_A) + \text{c.c.}\right) - |A|^2 \mathcal{N}(y_A, \bar{y}_A) + Ef$$

We choose:

$$y_{1/2} = \delta w_\delta + \left(A^2 e^{2i\omega_0 t} y_{AA} + \text{c.c.}\right) + |A|^2 y_{A\bar{A}} + E y_E$$

We have:

$$\mathcal{N}_{w_0} y_\delta \quad + \mathcal{L} y_\delta = \mathcal{M} w_0$$

$$2i\omega_0 \mathcal{B} y_{AA} + \mathcal{N}_{w_0} y_{AA} + \mathcal{L} y_{AA} = -\frac{1}{2}\mathcal{N}(y_A, y_A)$$

$$\mathcal{N}_{w_0} y_{A\bar{A}} + \mathcal{L} y_{A\bar{A}} = -\mathcal{N}(y_A, \bar{y}_A)$$

$$\mathcal{N}_{w_0} y_E + \mathcal{L} y_E \qquad = f$$

Order ϵ^1

PDE:

$$\mathcal{B}\partial_t y_1 + \mathcal{N}_{w_0} y_1 + \mathcal{L} y_1 = -\mathcal{B}\partial_\tau y_0 + \delta \mathcal{M} y_0 - \mathcal{N}\left(y_0, y_{1/2}\right)$$

$$= e^{i\omega_0 t}\left[-\frac{dA}{d\tau}\mathcal{B} y_A + \delta A \mathcal{M} y_A - \delta A \mathcal{N}(y_A, y_\delta) - A|A|^2 \mathcal{N}(y_A, y_{A\bar{A}})\right.$$

$$\left. - A|A|^2 \mathcal{N}(\bar{y}_A, y_{AA}) - AE\mathcal{N}(y_A, y_E)\right] + \text{c.c} + \cdots$$

Order ϵ^1

We kill the resonant terms to remove secular terms:

$$-\frac{dA}{d\tau} < \tilde{y}_A, \mathcal{B} y_A > + < \tilde{y}_A, \mathcal{M} y_A - \mathcal{N}(y_A, y_\delta) > \delta A$$
$$- < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) > A|A|^2 - < \tilde{y}_A, \mathcal{N}(y_A, y_E) > AE = 0$$

Where the adjoint global mode is:

$$-i\omega_0 \mathcal{B} \tilde{y}_A + \tilde{\mathcal{N}}_{w_0} \tilde{y}_A + \tilde{\mathcal{L}} \tilde{y}_A = 0$$

And has been scaled following:

$$< \tilde{y}_A, \mathcal{B} y_A > = 1$$

Hence:

$$\frac{dA}{d\tau} = \lambda \delta A - \mu A|A|^2 - \pi AE$$

With:

$$\lambda = < \tilde{y}_A, \mathcal{M} y_A > - < \tilde{y}_A, \mathcal{N}(y_A, y_\delta)$$
$$\mu = < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) >$$
$$\pi = < \tilde{y}_A, \mathcal{N}(y_A, y_E) >$$

The forced Navier-Stokes equations

Final solution:

$$w = \epsilon^{\frac{1}{2}} \left(\left(A(\tau) e^{i\omega_0 t} y_A + \text{c.c} \right) + \epsilon^{\frac{1}{2}} \left(\delta w_\delta + \left(A^2 e^{2i\omega_0 t} y_{AA} + \text{c.c.} \right) + |A|^2 y_{A\bar{A}} + E y_E \right) + \cdots \right)$$

Or:

$$w = \left(\tilde{A} e^{i\omega_0 t} y_A + \text{c.c} \right) + \tilde{\delta} w_\delta + \left(\tilde{A}^2 e^{2i\omega_0 t} y_{AA} + \text{c.c.} \right) + \left| \tilde{A} \right|^2 y_{A\bar{A}} + \tilde{E} y_E \cdots$$

With:

$$\frac{d\tilde{A}}{dt} = \lambda \tilde{\delta} \tilde{A} - \mu \tilde{A} \left| \tilde{A} \right|^2 - \pi \tilde{E} \tilde{A}$$
$$\pi = \langle \tilde{y}_A, \mathcal{N}_{y_A} y_E \rangle$$

Other expression for π:

$$\pi = \left\langle \left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}} \right)^{-1} \widetilde{\mathcal{N}_{y_A}} \tilde{y}_A, f \right\rangle$$

Proof:

$$\mathcal{N}_{w_0} y_E + \mathcal{L} y_E = f \Rightarrow y_E = \left(\mathcal{N}_{w_0} + \mathcal{L} \right)^{-1} f$$
$$\pi = \langle \tilde{y}_A, \mathcal{N}_{y_A} y_E \rangle = \langle \widetilde{\mathcal{N}_{y_A}} \tilde{y}_A, y_E \rangle = \left\langle \widetilde{\mathcal{N}_{y_A}} \tilde{y}_A, \left(\mathcal{N}_{w_0} + \mathcal{L} \right)^{-1} f \right\rangle = \left\langle \left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}} \right)^{-1} \widetilde{\mathcal{N}_{y_A}} \tilde{y}_A, f \right\rangle$$

Discussion: $|\tilde{A}| = O(1)$ and $\tilde{E} = 0$

$$\frac{d\tilde{A}}{dt} = \lambda\tilde{\delta}\tilde{A} - \mu\tilde{A}|\tilde{A}|^2$$
$$\omega_0 = 0.74$$
$$\lambda = 9.1 + 3.3i$$
$$\mu = 9.1 - 30.87i$$

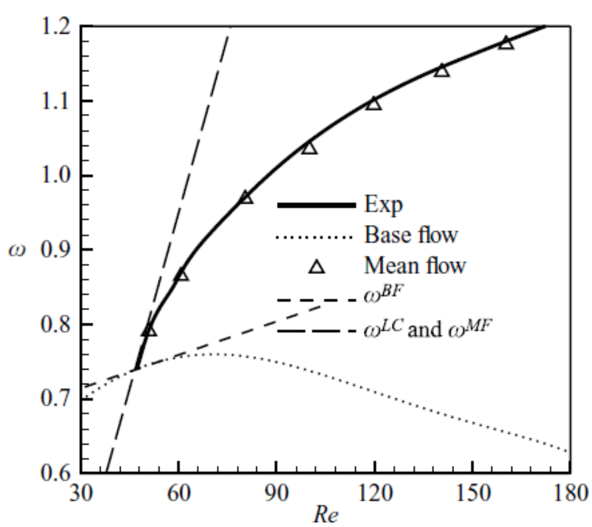
Solution on limit-cycle:

$$w = w_0 + \sqrt{\frac{\lambda_r}{\mu_r}\tilde{\delta}}\left[e^{i\left(\omega_0+\left(\lambda_i-\mu_i\frac{\lambda_r}{\mu_r}\right)\tilde{\delta}\right)t} y_A + \text{c.c}\right] + \cdots$$

Actual frequency on limit-cycle:

$$\omega = \omega_0 + \underbrace{\lambda_i\tilde{\delta}}_{\text{Linear}} \underbrace{-\mu_i\frac{\lambda_r}{\mu_r}\tilde{\delta}}_{\text{Non-linear interaction}}$$
$$= 0.74 + 3.3\tilde{\delta} + 30.87\tilde{\delta} \text{ (huge impact of nonlinearity)}$$

Discussion: $|\tilde{A}| = O(1)$ and $\tilde{E} = 0$



Sipp & Lebedev JFM 2007

Discussion: $|\tilde{A}| \ll 1$ and $\tilde{E} = O(1)$

With:

$$\frac{d\tilde{A}}{dt} = (\lambda\tilde{\delta} - \pi\tilde{E})\tilde{A}$$

Final solution:

$$w = \tilde{A}_0 e^{(i\omega_0 + \lambda\tilde{\delta} - \pi\tilde{E})t} y_A + \text{c.c}$$

New eigenvalue:

$$i\omega_0 + \lambda\tilde{\delta} - \pi\tilde{E}$$

Discussion: $|\tilde{A}| \ll 1$ and $\tilde{E} = O(1)$

Amplification rate:

$$\lambda_r\tilde{\delta} - \pi_r\tilde{E}$$

If $\pi_r > 0$, flow is stable if

$$\lambda_r\tilde{\delta} - \pi_r\tilde{E} < 0 \Rightarrow \tilde{E} > \frac{\lambda_r}{\pi_r}\tilde{\delta}$$

We have:

$$\pi_r = \left\langle \Re\left[\left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}}\right)^{-1}\widetilde{\mathcal{N}_{y_A}}\tilde{y}_A\right], f\right\rangle$$

Optimal choice for f:

$$f = \Re\left[\left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}}\right)^{-1}\widetilde{\mathcal{N}_{y_A}}\tilde{y}_A\right]$$

Frequency:

$$\omega_0 + \lambda_i\tilde{\delta} - \pi_i\tilde{E}$$

$$\pi_i = -\left\langle \Im\left[\left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}}\right)^{-1}\widetilde{\mathcal{N}_{y_A}}\tilde{y}_A\right], f\right\rangle$$

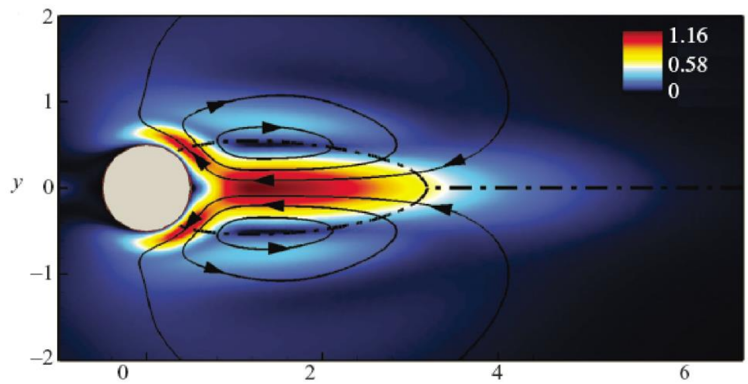
Optimal choice for f to modify frequency :

$$f = \Im\left[\left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}}\right)^{-1}\widetilde{\mathcal{N}_{y_A}}\tilde{y}_A\right]$$

The forced Navier-Stokes equations

Optimal choice for stabilization:

$$\Re\left[\left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}}\right)^{-1} \widetilde{\mathcal{N}_{y_A}} \tilde{y}_A\right]$$



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Results for cylinder flow

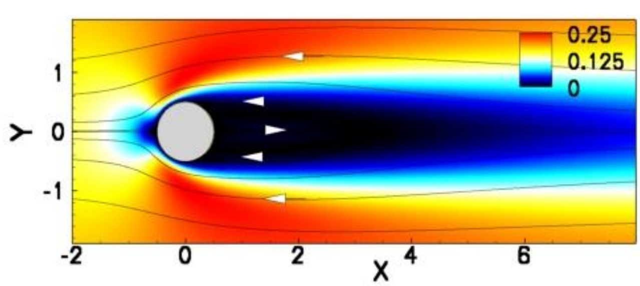
Modelling of small control cylinder (pure drag):

$$f = -u_0$$

Resulting amplification rate:

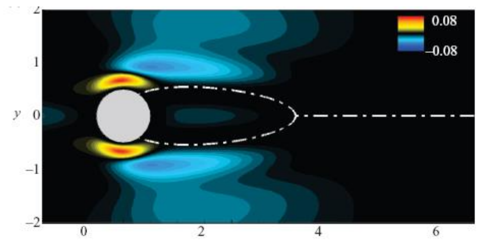
$$\lambda_r \tilde{\delta} - \pi_r \tilde{E} = \lambda_r \tilde{\delta} + \left\langle \Re\left[\left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}}\right)^{-1} \widetilde{\mathcal{N}_{y_A}} \tilde{y}_A\right], u_0 \right\rangle \tilde{E}$$

$-u_0$

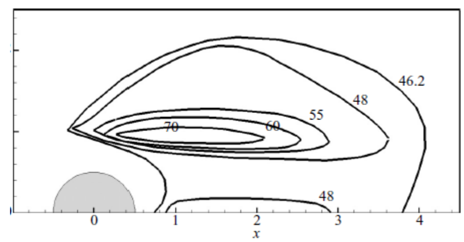


Results for cylinder flow

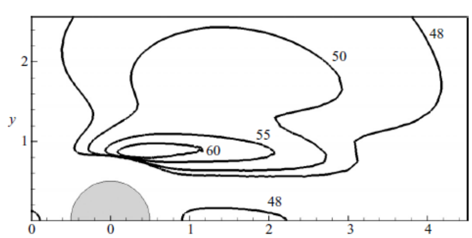
$$\left\langle \Re\left[\left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}}\right)^{-1} \widetilde{\mathcal{N}_{y_A}} \tilde{y}_A\right], u_0 \right\rangle$$



Experimental results
(Strykowski et al. 1990)

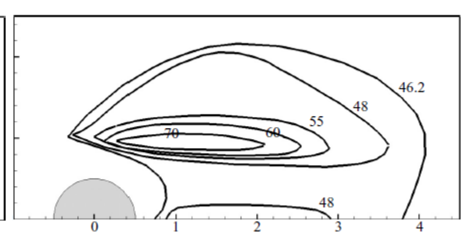


Results for cylinder flow



Theoretical results

$\tilde{E}$ is chosen according to the drag coefficient of a cylinder at small Reynolds



Experimental results

Strykowski et al. 1990

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- Vortex shedding in cylinder flow at $Re = 100$
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The forced Navier-Stokes equations

Forced Navier-Stokes equations with viscosity $\nu = \nu_c - \tilde{\delta}$:

$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = \tilde{\delta}\mathcal{M}w + \left(\tilde{E}e^{i\omega_f t} f + \text{c.c}\right)$$

$$w = \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} -\nu_c \Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}, \mathcal{M} = \begin{pmatrix} -\Delta & 0 \\ 0 & 0 \end{pmatrix}$$

Scalings for $\omega_f = \omega_0 + \tilde{\Omega}$:

$$\tilde{\delta} = \epsilon\delta, \qquad \delta = O(1)$$
$$w = w_0 + \epsilon^{1/2} y, \qquad \epsilon \ll 1,$$
$$\tilde{E} = \epsilon^{\frac{3}{2}} E, \qquad E = O(1)$$
$$\tilde{\Omega} = \epsilon\Omega, \qquad \Omega = O(1)$$

Base-flow:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0$$

Perturbation dynamics:

$$\mathcal{B}\partial_t y + \mathcal{N}_{w_0} y + \mathcal{L}y = \epsilon^{\frac{1}{2}}\delta\mathcal{M}w_0 + \epsilon\delta\mathcal{M}y - \frac{1}{2}\epsilon^{\frac{1}{2}}\mathcal{N}(y,y) + \epsilon\left(Ee^{i\omega_0 t}e^{i\epsilon\Omega t} f + \text{c.c}\right)$$

The forced Navier-Stokes equations

Expansion:

$$y = y_0(t, \tau = \epsilon t) + \epsilon^{1/2} y_{1/2}(t, \tau = \epsilon t) + \epsilon^1 y_1(t, \tau = \epsilon t) + \cdots$$

In particular:

$$\partial_t y = \partial_t y_0 + \epsilon^{1/2} \partial_t y_{1/2} + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots$$

The forced Navier-Stokes equations

$$\mathcal{B}\left(\partial_t y_0 + \epsilon^{1/2} \partial_t y_{1/2} + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots\right) + (\mathcal{N}_{w_0} + \mathcal{L})(y_0 + \epsilon^{1/2} y_{1/2} + \epsilon^1 y_1)$$
$$= \epsilon^{\frac{1}{2}} \delta\mathcal{M} w_0 + \epsilon \delta\mathcal{M} y_0 - \frac{1}{2} \epsilon^{\frac{1}{2}} \mathcal{N}(y_0, y_0) - \epsilon \mathcal{N}\left(y_0, y_{1/2}\right)$$
$$+ \epsilon\left(E e^{i\omega_0 t} e^{i\Omega\tau} f + \mathrm{c.c}\right)$$

$$\Rightarrow$$

$$\mathcal{B}\partial_t y_0 + \mathcal{N}_{w_0} y_0 + \mathcal{L} y_0 = 0$$

$$\mathcal{B}\partial_t y_{1/2} + \mathcal{N}_{w_0} y_{1/2} + \mathcal{L} y_{1/2} = \delta\mathcal{M} w_0 - \frac{1}{2} \mathcal{N}(y_0, y_0)$$

$$\mathcal{B}\partial_t y_1 + \mathcal{N}_{w_0} y_1 + \mathcal{L} y_1 = -\mathcal{B}\partial_\tau y_0 + \delta\mathcal{M} y_0 - \mathcal{N}\left(y_0, y_{\frac{1}{2}}\right) + \left(E e^{i\omega_0 t} e^{i\Omega\tau} f + \mathrm{c.c}\right)$$

Order ϵ^0

PDE:

$$\mathcal{B}\partial_t y_0 + \mathcal{N}_{w_0} y_0 + \mathcal{L} y_0 = 0$$

We choose:

$$y_0 = \left(A(\tau) e^{i\omega_0 t} y_A + \text{c.c}\right)$$

Order $\epsilon^{\frac{1}{2}}$

PDE:

$$\mathcal{B}\partial_t y_{1/2} + \mathcal{N}_{w_0} y_{1/2} + \mathcal{L} y_{1/2} = \delta \mathcal{M} w_0 - \frac{1}{2}\mathcal{N}(y_0, y_0)$$

$$= \delta \mathcal{M} w_0 + \left(-\frac{1}{2} A^2 e^{2i\omega_0 t} \mathcal{N}(y_A, y_A) + \text{c.c.}\right) - |A|^2 \mathcal{N}(y_A, \bar{y}_A)$$

We choose:

$$y_{1/2} = \delta w_\delta + \left(A^2 e^{2i\omega_0 t} y_{AA} + \text{c.c.}\right) + |A|^2 y_{A\bar{A}}$$

We have:

$$\mathcal{N}_{w_0} y_\delta \quad + \mathcal{L} y_\delta = \mathcal{M} w_0$$

$$2i\omega_0 \mathcal{B} y_{AA} + \mathcal{N}_{w_0} y_{AA} + \mathcal{L} y_{AA} = -\frac{1}{2}\mathcal{N}(y_A, y_A)$$

$$\mathcal{N}_{w_0} y_{A\bar{A}} + \mathcal{L} y_{A\bar{A}} = -\mathcal{N}(y_A, \bar{y}_A)$$

Order ϵ^1

PDE:

$$\mathcal{B}\partial_t y_1 + \mathcal{N}_{w_0} y_1 + \mathcal{L} y_1 = -\mathcal{B}\partial_\tau y_0 + \delta \mathcal{M} y_0 - \mathcal{N}(y_0, y_{1/2})$$

$$= e^{i\omega_0 t}\left[-\frac{dA}{d\tau}\mathcal{B}y_A + \delta A \mathcal{M} y_A - \delta A \mathcal{N}(y_A, y_\delta) - A|A|^2 \mathcal{N}(y_A, y_{A\bar{A}}) - A|A|^2 \mathcal{N}(\bar{y}_A, y_{AA}) + E e^{i\Omega\tau} f\right] + \text{c.c} + \cdots$$

Order ϵ^1

We kill the resonant terms to remove secular terms:

$$-\frac{dA}{d\tau} < \tilde{y}_A, \mathcal{B} y_A > + < \tilde{y}_A, \mathcal{M} y_A - \mathcal{N}(y_A, y_\delta) > \delta A$$
$$- < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) > A|A|^2 + < \tilde{y}_A, f > E e^{i\Omega\tau} = 0$$

Where the adjoint global mode is:

$$-i\omega_0 \mathcal{B} \tilde{y}_A + \tilde{\mathcal{N}}_{w_0} \tilde{y}_A + \tilde{\mathcal{L}} \tilde{y}_A = 0$$

And has been scaled following:

$$< \tilde{y}_A, \mathcal{B} y_A > = 1$$

Hence:

$$\frac{dA}{d\tau} = \lambda \delta A - \mu A|A|^2 + \pi E e^{i\Omega\tau}$$

With:

$$\lambda = < \tilde{y}_A, \mathcal{M} y_A > - < \tilde{y}_A, \mathcal{N}(y_A, y_\delta)$$
$$\mu = < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) >$$
$$\pi = < \tilde{y}_A, f >$$

The forced Navier-Stokes equations

Final solution:

$$w = \epsilon^{\frac{1}{2}}\left(\left(A(\tau)e^{i\omega_0 t}y_A + \text{c.c}\right) + \epsilon^{\frac{1}{2}}\left(\delta w_\delta + \left(A^2 e^{2i\omega_0 t}y_{AA} + \text{c.c.}\right) + |A|^2 y_{A\bar{A}}\right) + \cdots\right)$$

Or:

$$w = \left(\tilde{A}e^{i\omega_0 t}y_A + \text{c.c}\right) + \tilde{\delta}w_\delta + \left(\tilde{A}^2 e^{2i\omega_0 t}y_{AA} + \text{c.c.}\right) + \left|\tilde{A}\right|^2 y_{A\bar{A}} + \cdots$$

With:

$$\frac{d\tilde{A}}{dt} = \lambda\tilde{\delta}\tilde{A} - \mu\tilde{A}\left|\tilde{A}\right|^2 + \pi\tilde{E}e^{i\tilde{\Omega}t}$$

- It corresponds to an exact solution of the non-linear Navier-Stokes Eq.
- But no guarantee that it can be observed (stability? Unique-ness?)
- Valid near bifurcation threshold: $\left|\tilde{\delta}\right| \ll 1$

(Q4)

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- The Van der Pol oscillator: a model problem
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  - ➢ Two times-scales approach
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- The Ginzburg-Landau eq.
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  - ➢ Forcing with perturbation of linear operator ($\omega_f = 0$)
  - ➢ Forcing with ω_f close to ω_0
- The forced Navier-Stokes eq. with cylinder flow
  - ➢ Forcing with ω_f close to 0
  - ➢ Forcing with ω_f close to ω_0
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The forced Navier-Stokes equations

Forced Navier-Stokes equations with viscosity $\nu = \nu_c - \tilde{\delta}$:

$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = \tilde{\delta}\mathcal{M}w + \left(\tilde{E}e^{i\omega_f t}f + \text{c.c}\right)$$

$$w = \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} -\nu_c \Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}, \mathcal{M} = \begin{pmatrix} -\Delta & 0 \\ 0 & 0 \end{pmatrix}$$

Scalings for $\omega_f \neq \omega_0$:

$$\tilde{\delta} = \epsilon\delta, \qquad \delta = O(1)$$

$$w = w_0 + \epsilon^{1/2}y, \qquad \epsilon \ll 1,$$

$$\tilde{E} = \epsilon^{\frac{1}{2}}E, \qquad E = O(1)$$

Base-flow:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0$$

Perturbation dynamics:

$$\mathcal{B}\partial_t y + \mathcal{N}_{w_0}y + \mathcal{L}y = \epsilon^{\frac{1}{2}}\delta\mathcal{M}w_0 + \epsilon\delta\mathcal{M}y - \frac{1}{2}\epsilon^{\frac{1}{2}}\mathcal{N}(y,y) + \left(Ee^{i\omega_f t}f + \text{c.c}\right)$$

The forced Navier-Stokes equations

Expansion:

$$y = y_0(t, \tau = \epsilon t) + \epsilon^{1/2}y_{1/2}(t, \tau = \epsilon t) + \epsilon^1 y_1(t, \tau = \epsilon t) + \cdots$$

In particular:

$$\partial_t y = \partial_t y_0 + \epsilon^{1/2}\partial_t y_{1/2} + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots$$

The forced Navier-Stokes equations

$$\mathcal{B}\big(\partial_t y_0 + \epsilon^{1/2}\partial_t y_{1/2} + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots\big) + (\mathcal{N}_{w_0} + \mathcal{L})(y_0 + \epsilon^{1/2} y_{1/2} + \epsilon^1 y_1)$$
$$= \epsilon^{\frac{1}{2}}\delta\mathcal{M}w_0 + \epsilon\delta\mathcal{M}y_0 - \frac{1}{2}\epsilon^{\frac{1}{2}}\mathcal{N}(y_0, y_0) - \epsilon\mathcal{N}\big(y_0, y_{1/2}\big) + \big(Ee^{i\omega_f t}f + \text{c.c}\big)$$
$$\Rightarrow$$
$$\mathcal{B}\partial_t y_0 + \mathcal{N}_{w_0}y_0 + \mathcal{L}y_0 = \big(Ee^{i\omega_f t}f + \text{c.c}\big)$$
$$\mathcal{B}\partial_t y_{1/2} + \mathcal{N}_{w_0}y_{1/2} + \mathcal{L}y_{1/2} = \delta\mathcal{M}w_0 - \frac{1}{2}\mathcal{N}(y_0, y_0)$$
$$\mathcal{B}\partial_t y_1 + \mathcal{N}_{w_0}y_1 + \mathcal{L}y_1 = -\mathcal{B}\partial_\tau y_0 + \delta\mathcal{M}y_0 - \mathcal{N}\big(y_0, y_{1/2}\big)$$

Order ϵ^0

PDE:

$$\mathcal{B}\partial_t y_0 + \mathcal{N}_{w_0}y_0 + \mathcal{L}y_0 = \big(Ee^{i\omega_f t}f + \text{c.c}\big)$$

We choose:

$$y_0 = \big(A(\tau)e^{i\omega_0 t}y_A + \text{c.c}\big) + \big(Ee^{i\omega_f t}y_E + \text{c.c}\big)$$

Hence:

$$Ae^{i\omega_0 t}\underbrace{\big(i\omega_0\mathcal{B}y_A + \mathcal{N}_{w_0}y_A + \mathcal{L}y_A\big)}_{0} + Ee^{i\omega_f t}\big(i\omega_f\mathcal{B}y_E + \mathcal{N}_{w_0}y_E + \mathcal{L}y_E\big) + \text{c.c.} = Ee^{i\omega_f t}f + \text{c.c}$$

which leads to:

$$i\omega_f\mathcal{B}y_E + \mathcal{N}_{w_0}y_E + \mathcal{L}y_E = f$$

This linear problem can be solved because ω_f is not an eigenvalue of the Jacobian.

Order $\epsilon^{\frac{1}{2}}$

PDE:

$$\mathcal{B}\partial_t y_{1/2} + \mathcal{N}_{w_0} y_{1/2} + \mathcal{L} y_{1/2} = \delta\mathcal{M} w_0 - \frac{1}{2}\mathcal{N}(y_0, y_0)$$

$$= \delta\mathcal{M} w_0 + \left(-\frac{1}{2}A^2 e^{2i\omega_0 t}\mathcal{N}(y_A, y_A) + \text{c.c.}\right) - |A|^2\mathcal{N}(y_A, \bar{y}_A) - |E|^2\mathcal{N}(y_E, \bar{y}_E)$$

$$+ \left(-\frac{1}{2}E^2 e^{2i\omega_f t}\mathcal{N}(y_E, y_E) + \text{c.c.}\right) + \left(-AE e^{i(\omega_0+\omega_f)t}\mathcal{N}(y_A, y_E) + \text{c.c.}\right)$$

$$+ \left(-A\bar{E} e^{i(\omega_0-\omega_f)t}\mathcal{N}(y_A, \bar{y}_E) + \text{c.c.}\right)$$

We choose:

$$y_{1/2} = \delta w_\delta + \left(A^2 e^{2i\omega_0 t} y_{AA} + \text{c.c.}\right) + |A|^2 y_{A\bar{A}} + \left(E^2 e^{2i\omega_f t} y_{EE} + \text{c.c.}\right) + |E|^2 y_{E\bar{E}}$$

$$+ \left(AE e^{i(\omega_0+\omega_f)t} y_{AE} + \text{c.c.}\right) + \left(A\bar{E} e^{i(\omega_0-\omega_f)t} y_{A\bar{E}} + \text{c.c.}\right)$$

Order $\epsilon^{\frac{1}{2}}$

We have:

$$\mathcal{N}_{w_0} y_\delta + \mathcal{L} y_\delta = \mathcal{M} w_0$$

$$2i\omega_0 \mathcal{B} y_{AA} + \mathcal{N}_{w_0} y_{AA} + \mathcal{L} y_{AA} = -\frac{1}{2}\mathcal{N}(y_A, y_A)$$

$$\mathcal{N}_{w_0} y_{A\bar{A}} + \mathcal{L} y_{A\bar{A}} = -\mathcal{N}(y_A, \bar{y}_A)$$

$$2i\omega_f \mathcal{B} y_{EE} + \mathcal{N}_{w_0} y_{EE} + \mathcal{L} y_{EE} = -\frac{1}{2}\mathcal{N}(y_E, y_E)$$

$$\mathcal{N}_{w_0} y_{E\bar{E}} + \mathcal{L} y_{E\bar{E}} = -\mathcal{N}(y_E, \bar{y}_E)$$

$$2i(\omega_0 + \omega_f)\mathcal{B} y_{AE} + \mathcal{N}_{w_0} y_{AE} + \mathcal{L} y_{AE} = -\mathcal{N}(y_A, y_E)$$

$$2i(\omega_0 - \omega_f)\mathcal{B} y_{A\bar{E}} + \mathcal{N}_{w_0} y_{A\bar{E}} + \mathcal{L} y_{A\bar{E}} = -\mathcal{N}(y_A, \bar{y}_E)$$

Order ϵ^1

PDE:

$$\mathcal{B}\partial_t y_1 + \mathcal{N}_{w_0} y_1 + \mathcal{L} y_1 = -\mathcal{B}\partial_\tau y_0 + \delta\mathcal{M} y_0 - \mathcal{N}\left(y_0, y_{1/2}\right)$$

$$= e^{i\omega_0 t}\Big[-\frac{dA}{d\tau}\mathcal{B}y_A + \delta A\mathcal{M}y_A - \delta A\mathcal{N}(y_A, y_\delta) - A|A|^2\mathcal{N}(y_A, y_{A\bar{A}})$$
$$- A|A|^2\mathcal{N}(\bar{y}_A, y_{AA}) - A|E|^2\mathcal{N}(y_A, y_{E\bar{E}}) - A|E|^2\mathcal{N}(y_{\bar{E}}, y_{AE})$$
$$- A|E|^2\mathcal{N}(y_E, y_{A\bar{E}})\Big] + \text{c.c} + \cdots$$

Order ϵ^1

We kill the resonant terms to remove secular terms:

$$-\frac{dA}{d\tau} < \tilde{y}_A, \mathcal{B}y_A > + < \tilde{y}_A, \mathcal{M}y_A - \mathcal{N}(y_A, y_\delta) > \delta A$$
$$- < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) > A|A|^2 -$$
$$< \tilde{y}_A, \mathcal{N}(y_A, y_{E\bar{E}}) + \mathcal{N}(y_{\bar{E}}, y_{AE}) + \mathcal{N}(y_E, y_{A\bar{E}}) > A|E|^2 = 0$$

Where the adjoint global mode is:

$$-i\omega_0\mathcal{B}\tilde{y}_A + \tilde{\mathcal{N}}_{w_0}\tilde{y}_A + \tilde{\mathcal{L}}\tilde{y}_A = 0$$

And has been scaled following:

$$< \tilde{y}_A, \mathcal{B}y_A > = 1$$

Hence:

$$\frac{dA}{d\tau} = \lambda\delta A - \mu A|A|^2 - \pi A|E|^2$$

With:

$$\lambda = < \tilde{y}_A, \mathcal{M}y_A > - < \tilde{y}_A, \mathcal{N}(y_A, y_\delta)$$
$$\mu = < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) >$$
$$\pi = < \tilde{y}_A, \mathcal{N}(y_A, y_{E\bar{E}}) + \mathcal{N}(y_{\bar{E}}, y_{AE}) + \mathcal{N}(y_E, y_{A\bar{E}}) >$$

The forced Navier-Stokes equations

Final solution:

$$w = \epsilon^{\frac{1}{2}}\Big(\big(A(\tau)e^{i\omega_0 t}y_A + \text{c.c}\big) + \big(Ee^{i\omega_f t}y_E + \text{c.c}\big) + \epsilon^{\frac{1}{2}}\Big(\delta w_\delta + \big(A^2e^{2i\omega_0 t}y_{AA} + \text{c.c.}\big) + |A|^2 y_{A\bar{A}} + \big(E^2e^{2i\omega_f t}y_{EE} + \text{c.c.}\big) + |E|^2 y_{E\bar{E}} + \Big(AEe^{i(\omega_0+\omega_f)t}y_{AE} + \text{c.c.}\Big) + \Big(A\bar{E}e^{i(\omega_0-\omega_f)t}y_{A\bar{E}} + \text{c.c.}\Big)\Big) + \cdots\Big)$$

Or:

$$w = \big(\tilde{A}e^{i\omega_0 t}y_A + \text{c.c}\big) + \big(\tilde{E}e^{i\omega_f t}y_E + \text{c.c}\big) + \tilde{\delta}w_\delta + \big(\tilde{A}^2e^{2i\omega_0 t}y_{AA} + \text{c.c.}\big) + \big|\tilde{A}\big|^2 y_{A\bar{A}} + \big(\tilde{E}^2e^{2i\omega_f t}y_{EE} + \text{c.c.}\big) + \big|\tilde{E}\big|^2 y_{E\bar{E}} + \Big(\tilde{A}\tilde{E}e^{i(\omega_0+\omega_f)t}y_{AE} + \text{c.c.}\Big) + \Big(\tilde{A}\bar{\tilde{E}}e^{i(\omega_0-\omega_f)t}y_{A\bar{E}} + \text{c.c.}\Big) + \cdots$$

With:

$$\frac{d\tilde{A}}{dt} = \lambda\tilde{\delta}\tilde{A} - \mu\tilde{A}\big|\tilde{A}\big|^2 - \pi\tilde{A}\big|\tilde{E}\big|^2 \qquad \text{(Q4)}$$

- It corresponds to an exact solution of the non-linear Navier-Stokes Eq.
- But no guarantee that it can be observed (stability? Unique-ness?)
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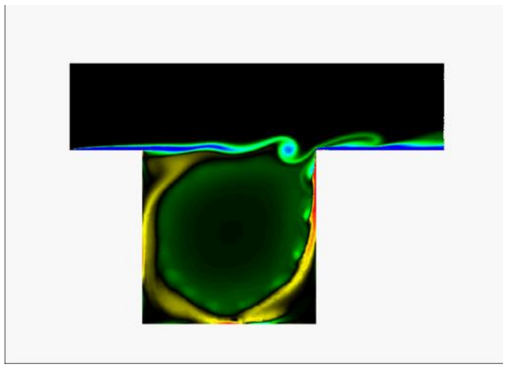
Configuration

Open cavity flow: 2D laminar transitional regime



Re_c — 5396

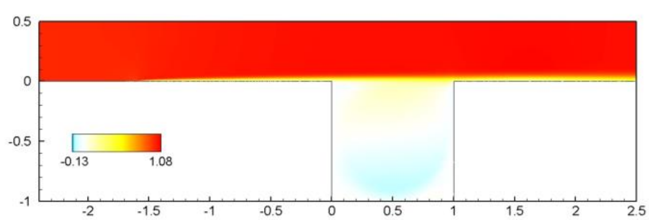
Re_{DNS} — 6250

Re

Base-flow

Order ϵ^0: base-flow w_0

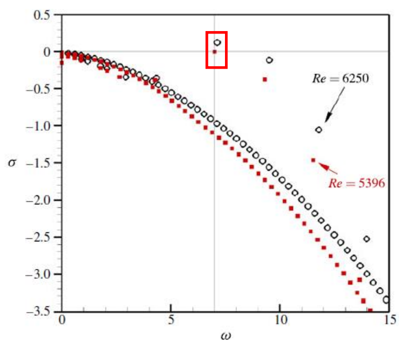
$Re = 5396$

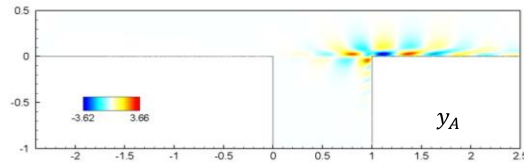


Streamwise velocity

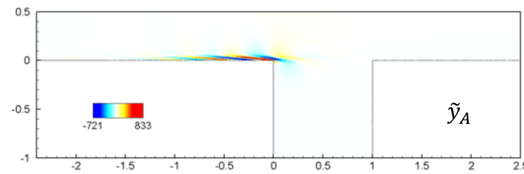
Linear dynamics

$Re = 5396, \omega_0 = 7.0$





Streamwise velocity



Amplitude equation

Supercritical Hopf bifurcation:

$$\frac{d\tilde{A}}{dt} = \lambda\tilde{\delta}\tilde{A} - \mu\tilde{A}\left|\tilde{A}\right|^2$$

$$\omega_0 = 7.0$$

$$\lambda = 4689 + 4702i$$

$$\mu = 3940 - 142i$$

Actual frequency on limit-cycle:

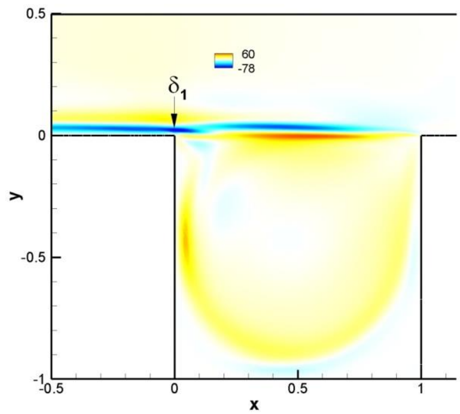
$$\omega = \omega_0 + \underbrace{\lambda_i\tilde{\delta}}_{\text{Linear}} \quad \underbrace{-\mu_i\frac{\lambda_r}{\mu_r}\tilde{\delta}}_{\text{Non-linear interaction}}$$

$$= 7.0 + 4702\tilde{\delta} + 168\tilde{\delta}\ \text{(small impact of nonlinearity)}$$

Amplitude equation with control
Steady forcing $\boldsymbol{\omega_f = 0}$ (small control cylinder)

Control cylinder: $f = -u_0$

Induced variation of amplification rate: $\left\langle \Re\left[\left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}}\right)^{-1} \widetilde{\mathcal{N}_{y_A}} \tilde{y}_A\right], u_0 \right\rangle$

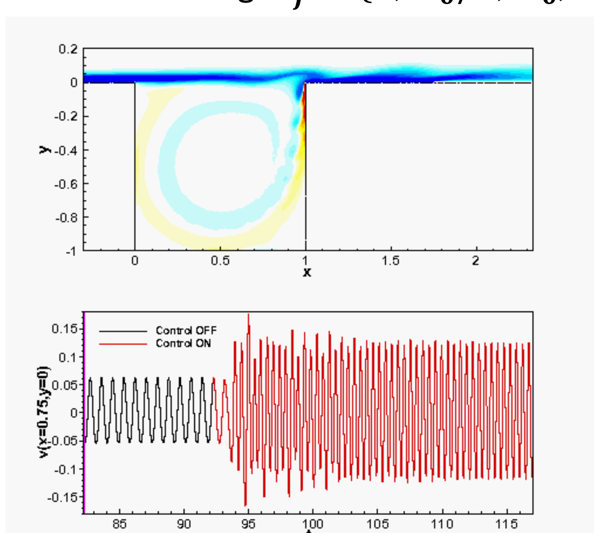


Amplitude equation with control
Non-resonant forcing $\boldsymbol{\omega_f \neq \{0, \omega_0/2, \omega_0, \cdots\}}$

$Re = 6250$

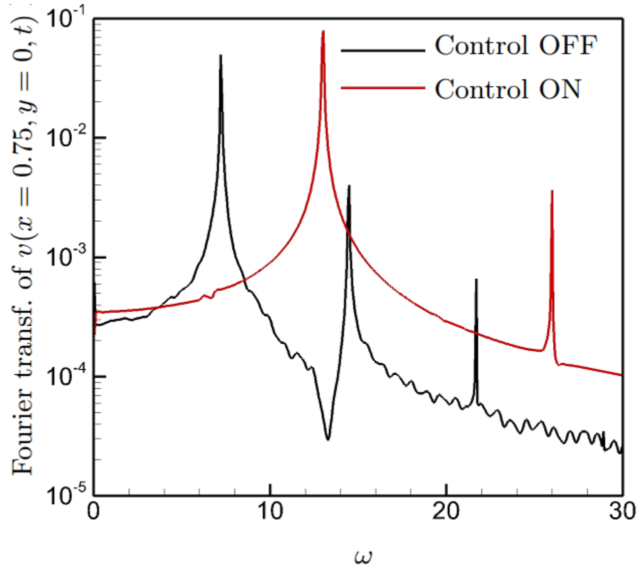
$\omega_f = 13$

$\tilde{E} = \tilde{E}_{STAB}$



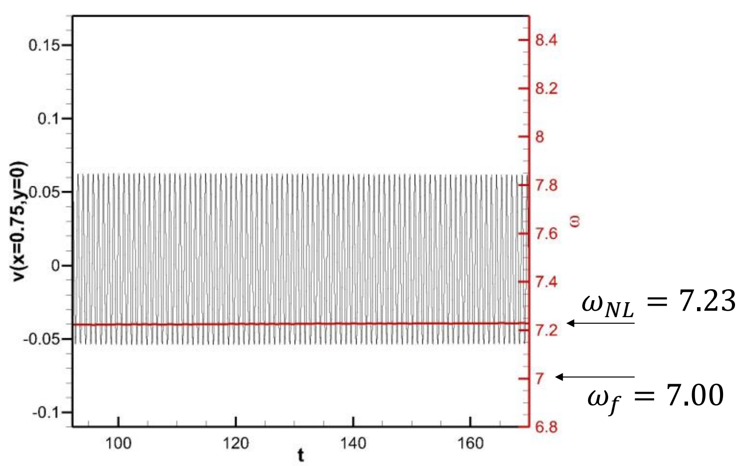
Amplitude equation with control
Non-resonant forcing $\boldsymbol{\omega_f} \neq \{\mathbf{0}, \boldsymbol{\omega_0}/\mathbf{2}, \boldsymbol{\omega_0}, \cdots\}$

$\omega_f = 13$



Amplitude equation with control
Resonant forcing $\boldsymbol{\omega_f} \approx \boldsymbol{\omega_0}$

$\tilde{E} = 0$



Amplitude equation with control

Resonant forcing $\boldsymbol{\omega_f \approx \omega_0}$

$\tilde{E} = 2\tilde{E}_{\text{Locking}}$